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Note

Existence Theorems for Sperner Families

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1. INTRODUCTION

Let $\mathscr{S}$ be a finite family of finite sets. We call $\mathscr{S}$ a Sperner family, or a clutter, if $X \not\subset Y$ for any two distinct sets X, Y of $\mathscr{S}$. The parameters of $\mathscr{S}$ are the numbers $p_0, p_1, \ldots,$ where p_i is the number of sets of cardinality i in $\mathscr{S}$. The sequence $p_0, p_1, \ldots,$ is of course finite. Also, a Sperner family consists of only the empty set $\varnothing$ if $p_0 > 0$, so we usually assume $p_0 = 0$. We write $[i, j]$ for the interval of integers i, $i + 1, \ldots, j$ which is $\varnothing$ when $i > j$.

This paper was motivated by a conjecture [2] of D. Kleitman and E.C. Milner which we prove true as

THEOREM 1. *Let $\mathscr{S}$ be a Sperner family of subsets of $[1, s]$. If $p_0, p_1, \ldots, p_s$ are the parameters of $\mathscr{S}$, then there is a Sperner family $\mathscr{Q}$ on $[1, s]$ with parameters $q_0, q_1, \ldots, q_s$, where $q_i = 0$ for $0 \leqslant i < \frac{1}{2}s$, and $q_i = p_{s-i} + p_i$ for $\frac{1}{2}s < i \leqslant s$, and when s is even $q_{\frac{1}{2}s} = p_{\frac{1}{2}s}$.*

In order to prove the theorem we needed a special class of Sperner families. These we were able to derive from the work of J. B. Kruskal [3], and we describe them in Section 5. They also enabled us to obtain

THEOREM 2. *Let $p_0, p_1, \ldots, p_g$ be nonnegative integers with $p = 0 < p_g$. Then the least integer s such that there exists a Sperner family on $[1, s]$ with parameters $p_0, p_1, \ldots, p_g$ is given by*

$$s = p_1 + K_2(p_2 + K_3(p_3 + \cdots + K_g(p_g) \cdots)), \tag{1}$$

where K denotes Kruskal's function defined in (2) *and* (3) *below.*

We were unable to deduce Theorem 1 from Theorem 2. Applications of Theorem 1 are mentioned in [2].

2. An Ordering Relation

For convenience we let all our sets be subsets of $\{1, 2, \ldots\}$. If X, Y are two distinct sets, we write $X < Y$ if the largest integer in the symmetric difference $(X \backslash Y) \cup (Y \backslash X)$ is in Y. Let $\ell \geqslant 1$ and $\mathscr{L}$ denote the family of all sets of cardinality ℓ. The ordering makes $\mathscr{L}$ into a sequence $L_1, L_2, \ldots$, which when $l = 3$ starts

$$\begin{array}{lcccccc}
L_1 = & 1 & 2 & 3 & \cdot & \cdot & \cdot \\
L_2 = & 1 & 2 & \cdot & 4 & \cdot & \cdot \\
L_3 = & 1 & \cdot & 3 & 4 & \cdot & \cdot \\
L_4 = & \cdot & 2 & 3 & 4 & \cdot & \cdot \\
L_5 = & 1 & 2 & \cdot & \cdot & 5 & \cdot \\
L_6 = & 1 & \cdot & 3 & \cdot & 5 & \cdot
\end{array}$$

In general, L_{i+1} is derivable from L_i by the

Successor Rule. *Given integers $1 \leqslant m \leqslant \ell$ and a set L of cardinality ℓ, let α be the least integer such that*

(i) $\alpha \in L$,
(ii) $\alpha + 1 \notin L$, *and*
(iii) $|L \cap [1, \alpha]| \geqslant 1 + \ell - m$.

Then the successor M of cardinality m of L is the set

$M = [1, \beta] \cup \{\alpha + 1\} \cup \{L \backslash [1, \alpha]\}$, *where* $\beta = |L \cap [1, \alpha]| - 1 - \ell + m$.

Notice that $L < M$, and when $m = \ell$, condition (iii) is redundant because it is implied by condition (i).

We now give three examples of using the successor rule. The way in which we have used it is indicated in the following diagram:

$$\begin{tikzcd}
V \arrow[rr] \arrow[d] & & W \arrow[d] \\
Y \arrow[r] & Z & X.
\end{tikzcd}$$

The sets V, W have cardinality 3 and the sets X, Y, Z have cardinality 2.

	Example 1	Example 2	Example 3
V	1 2 · 4 ·	1 · 3 4 ·	· 2 3 4 ·
W	1 · 3 4 ·	· 2 3 4 ·	1 2 · · 5
X	1 · · · 5	1 · · · 5	· · 3 · 5
Y	· · 3 4 ·	1 · · · 5	1 · · · 5
Z	1 · · · 5	· 2 · · 5	· 2 · · 5.

The examples have $X = Z$, $X < Z$, and $X > Z$, respectively, showing that the diagram is not commutative. They also show that distinct sets of one cardinality can have the same successor of another cardinality.

3. Kruskal–Katona Theorem

In this section we describe some results which were first found by Kruskal [3], then discovered independently by Katona [1].

Given positive integers ℓ, n, we can write n uniquely in the form

$$n = \binom{a_\ell}{\ell} + \binom{a_{\ell-1}}{\ell-1} + \cdots + \binom{a_t}{t}, \tag{2}$$

where $\ell \geqslant t \geqslant 1$ and $a_\ell > a_{\ell-1} > \cdots > a_t \geqslant t$. If $\mathscr{S}$ is a family of sets, then the deleting operator Δ deletes one element from a set of $\mathscr{S}$ in all possible ways to yield another family of sets $\Delta\mathscr{S}$.

Theorem 3. *If $\mathscr{S}$ is a family of n sets of cardinality ℓ, then $|\Delta\mathscr{S}| \geqslant K_\ell(n)$ where*

$$K_\ell(n) = \binom{a_\ell}{\ell-1} + \binom{a_{\ell-1}}{\ell-2} + \cdots + \binom{a_t}{t-1}. \tag{3}$$

In the next section we will define an example $\mathscr{S}$ which Kruskal called a cascade. The cascade has $|\Delta\mathscr{S}| = K_\ell(n)$ and thus shows that Theorem 3 is best possible. Repeated application of Theorem 3 gives the bound for $|\Delta^r\mathscr{S}|$. If we consider the case $\mathscr{S} = \mathscr{T} \cup \mathscr{U}$, where $\mathscr{T}, \mathscr{U}$ are disjoint families of n_1, n_2 sets respectively, we see that

$$K_\ell(n_1 + n_2) \leqslant K_\ell(n_1) + K_\ell(n_2) \tag{4}$$

for any nonnegative integers n_1, n_2.

4. Kruskal's Cascade

It is important to us that we can write out the cascade example explicitly–it is

$$\mathscr{S} = \mathscr{S}_\ell \cup \mathscr{S}_{\ell-1} \cup \cdots \cup \mathscr{S}_t, \tag{5}$$

where for $\ell \geqslant i \geqslant t$ we have

$$\mathscr{S}_i = \{X \cup Y_i : |X| = i, X \subset [1, a_i]\} \quad \text{and} \quad Y_i = \bigcup_{i<j\leqslant\ell} \{1 + a_j\}.$$

Notice that $Y_\ell = \varnothing$, $Y_{\ell-1} = \{1 + a_\ell\}$, $Y_{\ell-2} = \{1 + a_{\ell-1}\} \cup \{1 + a_\ell\}$, etc. Having expressed $\mathscr{S}$ in this way, we can immediately make a number of observations.

First, if $Z_i \in \mathscr{S}_i$ for $\ell \geqslant i \geqslant t$, then $Z_\ell < Z_{\ell-1} < \cdots < Z_t$. Thus the $\mathscr{S}_i$ are pairwise disjoint with $|\mathscr{S}_i| = \binom{a_i}{i}$, and so $|\mathscr{S}| = n$ by (2) and (5). After ordering $\mathscr{S}$, the last set in $\mathscr{S}$ is

$$\Omega = [1 - t + a_t, a_t] \cup Y_t,$$

because Ω is the last set in $\mathscr{S}_t$. If $|Z| = \ell$ and $Z < \Omega$, then $Z \in \mathscr{S}$. In other words, $\mathscr{S}$ consists of the first n sets of cardinality ℓ, thus $\mathscr{S} = \{L_1, L_2, \ldots, L_n\}$ in the notation of Section 2. Moreover, $\mathscr{S}$ can be generated by the successor rule with $\ell = m$.

Next we observe that

$$\begin{aligned}
\Delta\mathscr{S}_\ell &= \{X : |X| = \ell - 1, X \subset [1, a_\ell]\},\\
(\Delta\mathscr{S}_{\ell-1})\backslash\Delta\mathscr{S}_\ell &= \{X \cup Y_{\ell-1} : |X| = \ell - 2, X \subset [1, a_{\ell-1}]\},\\
(\Delta\mathscr{S}_{\ell-2})\backslash\Delta(\mathscr{S}_\ell \cup \mathscr{S}_{\ell-1}) &= \{X \cup Y_{\ell-2} : |X| = \ell - 3, X \subset [1, a_{\ell-2}]\},
\end{aligned}$$

and so on, the numbers of elements in these sets being $\binom{a_\ell}{\ell-1}$, $\binom{a_{\ell-1}}{\ell-2}$, $\binom{a_{\ell-2}}{\ell-3}$, and so on. In this way we see by (3) that $|\Delta\mathscr{S}| = K_\ell(n)$. The last set in $\Delta\mathscr{S}$ is obtained by deleting the first element from the last set Ω in $\mathscr{S}$, hence it is

$$\Psi = [2 - t + a_t, a_t] \cup Y_t.$$

If $|Z| = \ell - 1$ and $Z < \Psi$, then $Z \in \Delta\mathscr{S}$. Hence $\Delta\mathscr{S}$ consists of the first $K_\ell(n)$ sets of cardinality $\ell - 1$.

5. Special Sperner Families

We are now able to describe what we mean by the special Sperner family $\mathscr{F}$ having given parameters $p_0, p_1, \ldots, p_g$ with $p_0 = 0 < p_g$. If $\mathscr{F}_i^*$ denotes the collection of p_i sets of cardinality i in $\mathscr{F}$, then

$$\mathscr{F} = \mathscr{F}_1^* \cup \mathscr{F}_2^* \cup \cdots \cup \mathscr{F}_g^*$$

To simplify notation we put

$$q_g = p_g$$

and

$$q_i = p_i + K_{i+1}(q_{i+1}) \qquad \text{for} \quad i = g - 1, g - 2, \ldots, 1.$$

In particular, q_1 is the number in (1). Now $\mathscr{F}_g{}^*$ consists of the first q_g sets of cardinality g. As shown in the last section, $\Delta\mathscr{F}_g{}^*$ consists of the first $K_g(p_g) = q_{g-1} - p_{g-1}$ sets of cardinality $g - 1$, and the succeeding p_{g-1} sets we take as the second part $\mathscr{F}_{g-1}^*$ of $\mathscr{F}$. Thus $\mathscr{F}_{g-1}^* \cup \Delta\mathscr{F}_g{}^*$ consists of the first q_{g-1} sets of cardinality $g - 1$. Similarly, $\Delta(\mathscr{F}_{g-1}^* \cup \Delta\mathscr{F}_g{}^*)$ consists of the first $q_{g-2} - p_{g-2}$ sets of cardinality $g - 2$, and the next p_{g-2} sets we take for $\mathscr{F}_{g-2}^*$, and so on for $\mathscr{F}_{g-3}^*,\dots, \mathscr{F}_1{}^*$. It is clear that

$$\bigcup_{X\in\mathscr{F}} X = \mathscr{F}_1{}^* \cup \Delta(\mathscr{F}_2{}^* \cup \Delta(\cdots (\mathscr{F}_{g-1}^* \cup \Delta\mathscr{F}_g{}^*) \cdots)) = [1, q_1].$$

Another trivial fact, which will be important in the next section, is expressed as

LEMMA 1. *The special Sperner family with parameters p_0, p_1,..., p_{g-2}, q_{g-1} is $\mathscr{F}_1{}^* \cup \mathscr{F}_2{}^* \cup \cdots \cup \mathscr{F}_{g-1}^* \cup \Delta\mathscr{F}_g{}^*$.*

To prove Theorem 2 we now let $\mathscr{S}$ be any Sperner family with the given parameters $p_1,\dots, p_g$. For $1 \leqslant i \leqslant g$ we let $\mathscr{S}_i{}^*$ be the subfamily of p_i sets of cardinality i in $\mathscr{S}$. Then by Theorem 3 we have

$$| \Delta\mathscr{S}_g{}^* | \geqslant K_g(p_g),$$

so because $\mathscr{S}_{g-1}^*$ and $\Delta\mathscr{S}_g{}^*$ are disjoint,

$$| \mathscr{S}_{g-1}^* \cup \Delta\mathscr{S}_g{}^* | = | \mathscr{S}_{g-1}^* | + | \Delta\mathscr{S}_g{}^* | \geqslant q_{g-1}.$$

Since when ℓ is fixed $K_\ell(n)$ obviously does not decrease as n increases, applying Theorem 3 again we get

$$\begin{aligned}
&| \mathscr{S}_{g-2}^* \cup \Delta(\mathscr{S}_{g-1}^* \cup \Delta\mathscr{S}_g{}^*)| \\
&\quad = | \mathscr{S}_{g-2}^* | + | \Delta(\mathscr{S}_{g-1}^* \cup \Delta\mathscr{S}_g{}^*)| \\
&\quad \geqslant p_{g-2} + K_{g-1}(| \mathscr{S}_{g-1}^* \cup \Delta\mathscr{S}_g{}^* |) \geqslant p_{g-2} + K_{g-1}(q_{g-1}) = q_{g-2}.
\end{aligned}$$

Continuing in this way we find eventually that

$$\left| \bigcup_{X\in\mathscr{S}} X \right| = | \mathscr{S}_1{}^* \cup \Delta(\mathscr{S}_2{}^* \cup \Delta(\cdots (\mathscr{S}_{g-1}^* \cup \Delta\mathscr{S}_g{}^*) \cdots))| \geqslant q_1,$$

and Theorem 2 is proved. So also is

THEOREM 4. *For each Sperner family on* [1, *s*], *there is a special Sperner family on* [1, *s*] *with the same parameters.*

6. Proof of Theorem 1

Let s be a fixed positive integer. Given parameters $p_0, p_1, \ldots, p_s$, let h be the highest integer such that $p_{s-h} + p_h > 0$. If s is even and $h = \frac{1}{2}s$, the statement of Theorem 1 is trivially true with $\mathscr{Y} = \mathscr{S}$. If s is odd and $h = \frac{1}{2}(s+1)$, the statement is also true. For Sperner's theorem [2] says $\binom{s}{h} \geqslant p_0 + p_1 + \cdots + p_s = p_{h-1} + p_h$, so we can let $\mathscr{Y}$ consist of any $p_{h-1} + p_h$ sets of cardinality h. As an hypothesis for induction on h, we assume the statement is true for all families $\mathscr{S}$ having $\frac{1}{2}s \leqslant h < k$ and consider the case $h = k$.

Thus we suppose we have a family $\mathscr{S}$ with parameters $p_0, p_1, \ldots, p_s$ with $p_i = 0$ for $i < s - k$ and $i > k$, but with $p_{s-k} + p_k > 0$. For the moment we will assume s is even and $k \geqslant 3 + \frac{1}{2}s$. Starting from $\mathscr{S}$ we will define 6 Sperner families $\mathscr{R}_1, \ldots, \mathscr{R}_6$. The parameters of all these families are listed in the table, where for convenience we have put

$$a = p_k, \qquad b = p_{k-1}, \qquad c = p_{s-k+1}, \qquad d = p_{s-k},$$
$$K^a = K_k(a), \qquad K^d = K_k(d), \qquad K^e = K^a + K^d, \text{ and}$$
$$K^f = K_k(a+d) \leqslant K^e.$$

The inequality $K^f \leqslant K^e$ comes from (4).

Starting with $\mathscr{S}$ we apply Δ to the subfamily $\mathscr{S}_k{}^*$ of sets of $\mathscr{S}$ of cardinality k to get at least K^a new sets of cardinality $k-1$. Let $\mathscr{W}$ be any K^a of these sets, and $\mathscr{R}_1 = (\mathscr{S} \backslash \mathscr{S}_k{}^*) \cup \mathscr{W}$. To get $\mathscr{R}_2$ we take the complement of every set in $\mathscr{R}_1$. Then we get $\mathscr{R}_3$ from $\mathscr{R}_2$ in exactly the same way as we got $\mathscr{R}_1$ from $\mathscr{S}$. Since $\mathscr{R}_3$ has $p_k = p_{s-k} = 0$, by our induction hypothesis there is a family $\mathscr{R}_4$ with the parameters specified in the table.

Table of Parameter Values

	$\mathscr{S}$	$\mathscr{R}_1$	$\mathscr{R}_2$	$\mathscr{R}_3$	$\mathscr{R}_4$	$\mathscr{R}_5$	$\mathscr{R}_6$
$i > k$	0	0	0	0	0	0	0
$i = k$	a	0	d	0	0	0	$a+d$
$i = k-1$	b	$b + K^a$	c	$c + K^d$	$b+c+K^e$	$b+c+K^f$	$b+c$
$k-1 > i > \frac{1}{2}s$	p_i	p_i	p_{s-i}	p_{s-i}	$p_i + p_{s-i}$	$p_i + p_{s-i}$	$p_i + p_{s-i}$
$i = \frac{1}{2}s$	p_i	p_i	p_i	p_i	p_i	p_i	p_i
$\frac{1}{2}s > i > s-k+1$	p_i	p_i	p_{s-i}	p_{s-i}	0	0	0
$i = s-k+1$	c	c	$b + K^a$	$b + K^a$	0	0	0
$i = s-k$	d	d	0	0	0	0	0
$i < s-k$	0	0	0	0	0	0	0

In turn, because $K^e \geqslant K^f$ and we have the family $\mathscr{R}_4$, by Theorem 4 there is the special family $\mathscr{R}_5$, and hence by Lemma 1 the special family $\mathscr{R}_6$, each having parameters as specified.

The above argument will apply exactly as stated even if we did not have s even and $k \geqslant 3 + \frac{1}{2}s$, provided 1, 2, 3, or 4 lines as the case may be were deleted in the obvious way from the table. Our proof of Theorem 1 follows inductively.

Note Added in Proof (March 21, 1974). The authors would like to thank G. Katona for telling them that a generalization of Theorem 2 has been proved by G. F. Clements, A minimization problem concerning subsets of a finite set, *Discrete Math.* 4 (1973), 123–128.

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